Solution to Exercise 5[1]

1. (a) Note that

$$\begin{cases} \mathbf{x}_u = (1, 0, v) \\ \mathbf{x}_v = (0, 1, u) \end{cases}$$

Then, we have

$$\mathbf{x}_u \times \mathbf{x}_v = (-v, -u, 1) \neq \mathbf{0}$$

for any $(u, v) \in D$.
Thus, the parametrized surface is regular.

(b) Since

$$\begin{cases} \mathbf{x}_u = (-\cosh v \sin u, \cosh v \cos u, 0) \\ \mathbf{x}_v = (\sinh v \cos u, \sinh v \sin u, 1) \end{cases}$$

and

$$\mathbf{x}_u \times \mathbf{x}_v = (\cosh v \cos u, \cosh v \sin u, -\sinh v \cosh v)$$

Suppose that $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$, then we have

$$\begin{cases} \cosh v \cos u = 0 \\ \cosh v \sin u = 0 \\ \sinh v \cosh v = 0 \end{cases}$$

For the first two equations, since $\cosh v \neq 0$, so $\cos u = \sin u = 0$, contradiction!
Thus, the parametrized surface is regular.

(c) Since

$$\begin{cases} \mathbf{x}_u = (1 - u^2 + v^2, 2uv, 2u) \\ \mathbf{x}_v = (2uv, 1 - v^2 + u^2, -2v) \end{cases}$$

and

$$\mathbf{x}_u \times \mathbf{x}_v = -(u^2 + v^2 + 1)\left(2u, -2v, u^2 + v^2 - 1\right)$$

Suppose that $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$, then we have

$$\begin{cases} 2u = 0 \\ 2v = 0 \\ u^2 + v^2 - 1 = 0 \end{cases}$$

From the first two equations, we have $u = v = 0$ but $u^2 + v^2 - 1 = -1 \neq 0$, contradiction!
Thus, the parametrized surface is regular.

2. (a) Note that

$$\begin{cases} \mathbf{x}_u = (1, 0, f_u) \\ \mathbf{x}_v = (0, 1, f_v) \end{cases}$$

Then, we have

$$\mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1) \neq \mathbf{0}$$

for any $(u, v) \in D$.
Thus, the surface $\mathbf{x}(D)$ is a regular parametrized surface.

[1] If you have any problems or spot any typos, please contact me via **maxshung.math@gmail.com**

(b) (i) Parametrize S by $\mathbf{x}(u,v) = (u, v, u^2 - v^2)$, and it is a graphical surface. From (a), the surface is regular.

(ii) Parametrize S by $\mathbf{x}(u,v) = (u, v, u - v + 1)$, and it is a graphical surface. From (a), the surface is regular.

(c) From (a), the normal vector to the surface $\mathbf{x}(D)$ is $(-f_u, -f_v, 1)$.
Let $(x, y, z) \in \mathbf{x}(D)$, then

$$\begin{aligned}
\langle (x,y,z) - (u,v,f(u,v)), (-f_u, -f_v, 1)\rangle &= 0 \\
-f_u(x-u) - f_v(y-v) + (z - f(u,v)) &= 0 \\
z - f(u,v) &= f_u(x-u) + f_v(y-v) \\
z &= f(u,v) + f_u(x-u) + f_v(y-v)
\end{aligned}$$

and this gives us the tangent plane of the graphical surface at the point $p = \mathbf{x}(u,v)$.

(d) Parametrize the surface S by $\mathbf{x}(u,v) = (u, v, u^2 + v^2)$.
Then, we have $f(u,v) = u^2 + v^2$, and $f_u = 2u, f_v = 2v$.
From part (c), the tangent plane at the point $(4, -3, 25)$ is given by

$$\begin{aligned}
z &= f(4,-3) + \frac{\partial f}{\partial u}(4,-3)(x-4) + \frac{\partial f}{\partial v}(4,-3)(y-(-3)) \\
z &= 25 + 2(4)(x-4) + 2(-3)(y+3) \\
z &= 8x - 6y - 25
\end{aligned}$$

3. Let $x_0 \in S$, for any $y_0 \in S$.
Since S is connected, so there is a smooth curve $\alpha(s) : [0, \ell] \to S$ such that

$$\alpha(0) = x_0, \alpha(\ell) = y_0$$

So $\alpha'(s) \in T_{\alpha(s)}S$ for any $s \in [0, \ell]$.
By the assumption that all normal lines of S passes through $\mathbf{p}$, so

$$\begin{aligned}
(\mathbf{p} - \alpha(s)) &\perp T_{\alpha(s)}S \\
\langle \mathbf{p} - \alpha(s), \alpha'(s)\rangle &= 0 \\
\frac{d}{ds}\langle \mathbf{p} - \alpha(s), \mathbf{p} - \alpha(s)\rangle &= 0
\end{aligned}$$

So $\|\alpha(s) - \mathbf{p}\| = R$ for some positive constants R. (Note that $R > 0$ because $\alpha(s)$ contains not only one point), so

$$\|x_0 - \mathbf{p}\| = \|y_0 - \mathbf{p}\| = R$$

and thus S lies on a sphere since y_0 is arbitrary on S.

4. (a) Note that

$$\begin{cases} \mathbf{x}_u = (a\sinh u\cos v, a\sinh u \sin v, c\cosh u) \\ \mathbf{x}_v = (-a\cosh u \sin v, a\cosh u\cos v, 0) \end{cases}$$

Then we have

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle \\
&= a^2\sinh^2 u + c^2\cosh^2 u \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\
&= -a^2 \sinh u\cos u\cos v\sin v + a^2\sinh u\cosh u\sin v\cos v + 0 \\
&= 0 \\
&= \langle \mathbf{x}_v, \mathbf{x}_u\rangle \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle \\
&= a^2\cosh^2 u
\end{aligned}$$

Thus, the first fundamental form of $\mathbf{x}$ as a 2×2 matrix is

$$I = \begin{pmatrix} a^2 \sinh^2 u + c^2 \cosh^2 u & 0 \\ 0 & a^2 \cosh^2 u \end{pmatrix}$$

(b) Since

$$\begin{cases} \mathbf{y}_u = (a \cosh u \cos v, a \cosh u \sin v, c \sinh u) \\ \mathbf{y}_v = (-a \sinh u \sin v, a \sinh u \cos v, 0) \end{cases}$$

Then, we have

$$\begin{aligned} E &= \langle \mathbf{y}_u, \mathbf{y}_u \rangle \\ &= a^2 \cosh^2 u + c^2 \sinh^2 u \\ F &= \langle \mathbf{y}_u, \mathbf{y}_v \rangle \\ &= -a^2 \cosh u \sinh u \cos v \sin v + a^2 \cosh u \sinh u \sin v \cos v + 0 \\ &= 0 \\ &= \langle \mathbf{y}_v, \mathbf{y}_u \rangle \\ G &= \langle \mathbf{y}_v, \mathbf{y}_v \rangle \\ &= a^2 \sinh^2 u \end{aligned}$$

Thus, the first fundamental form of $\mathbf{y}$ as a 2×2 matrix is

$$I = \begin{pmatrix} a^2 \cosh^2 u + c^2 \sinh^2 u & 0 \\ 0 & a^2 \sinh^2 u \end{pmatrix}$$

5. (a) Using First principles, note that

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\ &= \lim_{h \to 0} \left(\frac{\mathbf{x}(u(t+h), v(t+h)) - \mathbf{x}(u(t), v(t+h))}{h} + \frac{\mathbf{x}(u(t), v(t+h)) - \mathbf{x}(u(t), v(t))}{h} \right) \\ &= \lim_{h \to 0} \left[\left(\frac{\mathbf{x}(u(t+h), v(t+h)) - \mathbf{x}(u(t), v(t+h))}{u(t+h) - u(t)} \cdot \frac{u(t+h) - u(t)}{h} \right) \right. \\ &\qquad \left. + \left(\frac{\mathbf{x}(u(t), v(t+h)) - \mathbf{x}(u(t), v(t))}{v(t+h) - v(t)} \cdot \frac{v(t+h) - v(t)}{h} \right) \right] \end{aligned}$$

Since $\mathbf{x}$ is differentiable, so $\mathbf{x}_u, \mathbf{x}_v$ exist for any (u, v).
Now, we consider $\mathbf{r}'(t)$ in the following situations:

- If $u(t)$ and $v(t)$ and constant functions, then

$$\begin{cases} u(t+h) - u(t) = 0 \\ v(t+h) - v(t) = 0 \end{cases}, \quad \forall t \in (a, b)$$

  Then $\mathbf{x}(u, v)$ is not a surface as $\mathbf{x}^{-1}(S) = \{(u_0, v_0)\}$ which is a point.
- If either $u(t)$ or $v(t)$ is a constant function, then

$$u(t+h) - u(t) = 0, \quad \text{or} \quad v(t+h) - v(t) = 0$$

  for any $t \in (a, b)$.
  Then, $\mathbf{x}(u, v)$ is not a surface as $\mathbf{x}^{-1}(S) = \{(u, v_0)\}$ or $\{(u_0, v)\}$ which are curves.

- If both $u(t)$ and $v(t)$ are non-constant functions, then we have $\mathbf{x}(u,v)$ is a surface as $\mathbf{x}^{-1}(S) = \{(u,v)\} \subseteq D$ which is a connected subset of $\mathbb{R}^2$, by definition of regular parametrized surface $\mathbf{x}$.

Therefore, we only consider the third case.
If both $u(t)$ and $v(t)$ are non-constant functions, then there exists $\delta, \delta^* > 0$ such that

$$|u(x) - u(t)| > 0 \text{ for any } x \in (t-\delta, t+\delta) \setminus \{t\}$$

and

$$|v(x) - v(t)| > 0 \text{ for any } x \in (t-\delta^*, t+\delta^*) \setminus \{t\}.$$

So now, we choose $\delta' = \min\{\delta, \delta^*\} > 0$ such that $|u(t+h)-u(t)| > 0, |v(t+h)-u(t)| > 0$ for any $h \in (-\delta', \delta') \setminus \{0\}$.
Putting back to the above limit formula, we have

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{u(t+h)\to u(t), h\to 0} \left(\frac{\mathbf{x}(u(t+h), v(t+h)) - \mathbf{x}(u(t), v(t+h))}{u(t+h) - u(t)} \right) \cdot \lim_{h\to 0} \frac{u(t+h) - u(t)}{h} \\
&\quad + \lim_{v(t+h)\to v(t), h\to 0} \left(\frac{\mathbf{x}(u(t), v(t+h)) - \mathbf{x}(u(t), v(t))}{v(t+h) - v(t)} \right) \cdot \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} \\
&= \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}
\end{aligned}$$

(b) From (a), note that

$$\begin{aligned}
\|\mathbf{r}'(t)\|^2 = \langle \mathbf{r}'(t), \mathbf{r}'(t) \rangle &= \left\langle \begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}, \begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \right\rangle \\
&= \left(\begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \right)^T \left(\begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \right) \\
&= \begin{pmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{pmatrix} \begin{pmatrix} \mathbf{x}_u \\ \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \\
&= \begin{pmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \\
&= \begin{pmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{pmatrix} I \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}
\end{aligned}$$

Thus, the arc length of $\mathbf{r}(t)$ is given by

$$\ell = \int_a^b \|\mathbf{r}'(t)\| dt = \int_a^b \sqrt{\begin{pmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{pmatrix} I \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}}\, dt,$$

where I is the first fundamental form of $\mathbf{x}(u,v)$

6. (a) Since

$$\begin{cases} \mathbf{x}_u = (3u^2\cos\theta, 3u^2\sin\theta, 1) \\ \mathbf{x}_\theta = (-u^3\sin\theta, u^3\cos\theta, 0) \end{cases}$$

Then, we have

$$\begin{aligned} E &= 1 + 9u^4 \\ F &= -3u^5\cos\theta\sin\theta + 3u^5\sin\theta\cos\theta + 0 \\ &= 0 \\ G &= u^6 \end{aligned}$$

The first fundamental form of $\mathbf{x}$ as a 2×2 matrix is

$$I = \begin{pmatrix} 1 + 9u^4 & 0 \\ 0 & u^6 \end{pmatrix}$$

and the required surface area is

$$\begin{aligned} \iint_D 1\,dA &= \int_0^{2\pi}\int_0^1 \sqrt{\det(I)}\,dud\theta \\ &= \int_0^{2\pi}\int_0^1 \sqrt{u^6(1+9u^4)}\,dud\theta \\ &= 2\pi\int_0^1 u^3\sqrt{1+9u^4}\,du \\ &= 2\pi\cdot\frac{1}{36}\int_0^1 (1+9u^4)^{\frac{1}{2}}\,d(1+9u^4) \\ &= \frac{\pi}{18}\left[\frac{(1+9u^4)^{\frac{3}{2}}}{3/2}\right]_{u=0}^{u=1} \\ &= \frac{\pi}{27}\left(10^{\frac{3}{2}} - 1\right) \end{aligned}$$

(b) Note that

$$\begin{cases} \mathbf{x}_u = (\cos v, \sin v, 0) \\ \mathbf{x}_v = (-u\sin v, u\cos v, 1) \end{cases}$$

Then, we have

$$\begin{aligned} E &= 1 \\ F &= -u\cos v\sin v + u\sin v\cos v + 0(1) \\ &= 0 \\ G &= 1 + u^2 \end{aligned}$$

The first fundamental form of $\mathbf{x}$ as a 2×2 matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 + u^2 \end{pmatrix}$$

and the required surface area is

$$
\begin{aligned}
\iint_D 1\,dA &= \int_0^{2\pi}\int_{-1}^{1}\sqrt{\det(I)}\,dudv \\
&= \int_0^{2\pi}\int_{-1}^{1}\sqrt{1+u^2}\,dudv \\
&= 2\pi\cdot 2\int_0^1 \sqrt{1+u^2}\,du \\
&= 2\pi\left(\sinh^{-1}(1)+\sqrt{2}\right) \\
&= 2\pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right)
\end{aligned}
$$

(c) Since

$$
\begin{cases}
\mathbf{x}_u = (-2(1+\cos v)\sin u, 2(1+\cos v)\cos u, 0) \\
\mathbf{x}_v = (-2\cos u\sin v, -2\sin u\sin v, 2\cos v)
\end{cases}
$$

Then, we have

$$
\begin{aligned}
E &= [2(1+\cos v)(\sin u)]^2 + [2(1+\cos v)\cos u]^2 + 0^2 \\
&= 4(1+\cos v)^2(\sin^2 u+\cos^2 u) \\
&= 4(1+\cos v)^2 \\
F &= 4(1+\cos v)\sin u\cos u\sin v - 4(1+\cos v)\sin u\cos u\sin v + 0(2\cos v) \\
&= 0 \\
G &= [-2\cos u\sin v]^2 + [-2\sin u\sin v]^2 + [2\cos v]^2 \\
&= 4\sin^2 v(\cos^2 u+\sin^2 u) + 4\cos^2 v \\
&= 4\sin^2 v + 4\cos^2 v \\
&= 4
\end{aligned}
$$

Thus, the first fundamental form of the horn torus under $\mathbf{x}$ as a 2×2 matrix is

$$
I = \begin{pmatrix} 4(1+\cos v)^2 & 0 \\ 0 & 4 \end{pmatrix}
$$

and the required surface area is

$$
\begin{aligned}
\iint_D 1\,dA &= \int_0^{2\pi}\int_0^1 \sqrt{4\cdot 4(1+\cos v)^2}\,dudv \\
&= \int_0^{2\pi} 4(1+\cos v)\cdot u\Big|_{u=0}^{u=1}\,dv \\
&= [4v+4\sin v]_{v=0}^{v=2\pi} \\
&= 8\pi
\end{aligned}
$$